HW 7 — Due: Not Due

Problem 1. Find the "perceived" frequency when we sample the following signals at sampling rate $f_s = 12$ [Sa/s]:

(a) the signal $\cos(2\pi(1,111,111)t)$

(b) the signal $\cos(2\pi(111,111)t)$

(c) the signal $e^{j2\pi(11,111)t}$

(d) the signal $e^{-j2\pi(1,111)t}$

Remark: In lecture, we defined the "perceived" frequency to be the frequency that `plotspect` "sees" the signal. Now that we learned about reconstruction, we can define the "perceived" frequency to be the frequency of the signal reconstructed by the "reconstruction equation". If the reconstructed signal is $\cos(2\pi \tilde{f}_0 t)$ where $\tilde{f}_0 \geq 0$, the perceived frequency is $\tilde{f}_0$. If the reconstructed signal is $e^{j 2\pi \tilde{f}_0 t}$, the perceived frequency is $\tilde{f}_0$. Note that for complex-exponential signal, the perceived frequency can be negative.
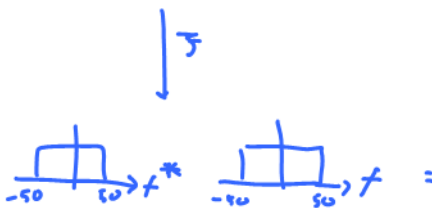
$2 \times f_{max}$ $\frac{1}{2 f_{max}}$

Problem 2. Determine the Nyquist sampling rate and the Nyquist sampling interval for the following signals:

(a) $\mathrm{sinc}(100\pi t)$
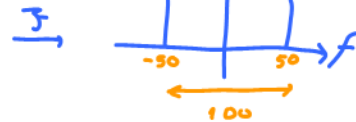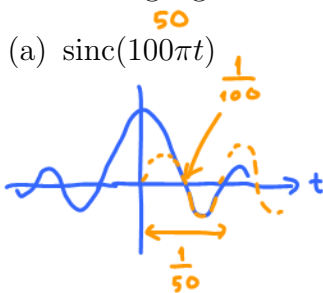
$f_{max} = 50$

$R_{Nyquist} = 2 \times 50 = 100$

$T_{Nyquist} = \frac{1}{100} = 0.01$

(b) $\mathrm{sinc}^2(100\pi t)$

$f_{max} = 100$

$R_{Nyquist} = 2 \times 100 = 200$

$T_{Nyquist} = \frac{1}{200} = 0.005$

(c) $\mathrm{sinc}(100\pi t) + \mathrm{sinc}(50\pi t)$

(d) $\mathrm{sinc}(100\pi t) + 3\,\mathrm{sinc}^2(60\pi t)$

  (e) $\mathrm{sinc}(50\pi t)\,\mathrm{sinc}(100\pi t)$

Remark: Recall that in our class, $\mathrm{sinc}(x) = \frac{\sin(x)}{x}$.

Problem 3. Consider a signal $g(t) = \mathrm{sinc}(\pi t)$.

  (a) Sketch its Fourier transform $G(f)$.

  (b) Find the Nyquist sampling rate.

  (c) The sequence of sampled values $g[n]$ is constructed from $g(t)$ by

$$g[n] = g\left(t\right)|_{t=nT_s}.$$

    Recall that the instantaneous sampled signal $g_\delta(t)$ is defined by

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g[n]\delta(t - nT_s)$$

    where T_s is the sampling interval.

(i) Let $T_s = 0.5$. Sketch the Fourier transform $G_\delta(f)$ of $g_\delta(t)$.

(ii) Let $T_s = 4/3$. Sketch the Fourier transform $G_\delta(f)$ of $g_\delta(t)$.

(d) Recall the reconstruction equation:

$$\hat{g}(t) = \sum_{n=-\infty}^{\infty} g[n] \operatorname{sinc}(\pi f_s(t - nT_s)). \qquad (7.1)$$

Note that we write $\hat{g}(t)$ instead of $g(t)$ to accommodate the case that the sampling rate is too low; in which case, the reconstructed signal is not the same as $g(t)$.

With $T_s = 1$,

(i) Find $g[n]$ for $n = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots$

(ii) Use the reconstruction equation to find $\hat{g}(t)$.

Extra Questions

Here are some optional questions for those who want more practice.

Problem 4. We can use `MATLAB` to study the effect of sampling in both time and frequency domain. Consider a bipolar square pulse periodic signal:

$$g(t) = \text{sgn}\left(\cos\left(2\pi f_0 t\right)\right) = \begin{cases} 1, & \cos\left(2\pi f_0 t\right) \geq 0, \\ -1, & \cos\left(2\pi f_0 t\right) < 0. \end{cases}$$

(a) When $f_0 = 10$ [Hz], plot $g(t)$ from $t = 0$ [s] to $t = 0.5$ [s].

(b) Note that the bipolar square pulse periodic signal is almost the same as our ON-OFF signal studied in lecture. There,

$$g_{\text{ON - OFF}}(t) = 1\left[\cos\left(2\pi f_0 t\right) \geq 0\right] = \begin{cases} 1, & \cos\left(2\pi f_0 t\right) \geq 0, \\ 0, & \cos\left(2\pi f_0 t\right) < 0. \end{cases}$$

The ON-OFF signal studied in lecture alternates between "0 and 1". Here, $g(t)$ alternates between "-1 and 1". Find the constant a and b such that

$$g(t) = a \times g_{\text{ON - OFF}}(t) + b.$$

(c) The (magnitude) spectrum of a periodic square wave can be found using:

```
% specsquare.m plot the spectrum of a square wave
close all
f0=10;                    % "(fundamental) frequency" of the square wave
EndTime=2;                % Will consider from time = 0 to EndTime
Ts=1/1000;                % sampling interval
t=0:Ts:(EndTime-Ts);      % create a time vector
g=sign(cos(2*pi*f0*t));   % square wave = sign of cos wave
plotspect(g,t)            % call plotspec to draw spectrum
```

The output of `specsquare.m` is shown in Figure 7.1. The top plot shows the first 2 seconds of $g(t)$ with fundamental frequency $f_0 = 10$ cycles per second. The bottom plot shows a series of spikes that define its frequency content.

In lecture, we showed that

$$g_{\text{ON - OFF}}(t) = 1\left[\cos\left(2\pi f_0 t\right) \geq 0\right]$$
$$= \frac{1}{2} + \frac{2}{\pi}\left(\cos 2\pi f_0 t - \frac{1}{3}\cos 2\pi 3 f_0 t + \frac{1}{5}\cos 2\pi 5 f_0 t - \frac{1}{7}\cos 2\pi 7 f_0 t + \dots\right).$$

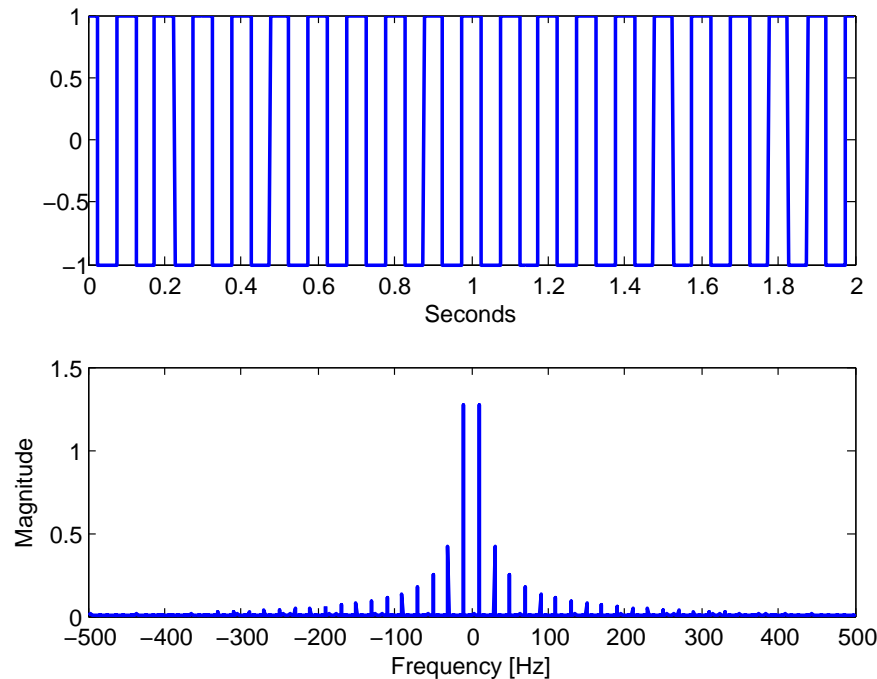(i) Use the relationship derived in part (b) to find the Fourier series expansion of $g(t)$.

Figure 7.1: Plots from `specsquare.m`

(ii) Use the Fourier series expansion in the previous part to double-check the lower plot in Figure 7.1.

    i. Do the spikes show up at the expected locations?

    ii. Note that, in the script, we consider the time between 0 to 2 [s]. Therefore, actually, we are not looking at the signal $g(t)$ from $-\infty$ to ∞. This time-limited view means that, in the frequency domain, we won't see the impulses but rather sinc pulses at the expected locations. (This is the same as seeing two sinc pulses instead of two impulses when looking at the Fourier transform of the cosine pulse.)

    In Figure 7.1, the height of the highest spikes in the center is about 1.2 or 1.3. Calculate (derive analytically) the exact hight of these spikes.

(d) In this part, we increase the value of f_0 in `specsquare.m` to 20, 40, 100, and 300 Hz while keeping the sampling interval at $T_s = 1/1000$. The magnitude spectrum plots (the lower subplots from `specsquare.m`) for these cases are shown in Figure 7.2.

Use the same analysis as in part c.(ii).i and the "folding technique" to explain (the aliasing effect in) each of the provided plots

Problem 5. Again, consider a signal $g(t) = \text{sinc}(\pi t)$. Let's test the reconstruction equation (7.1) by using MATLAB to plot $\hat{g}(t)$. Note that the sum in the reconstruction equation extends from $-\infty$ to $+\infty$. In MATLAB, we can not add that many terms. So, we need to stop at some n. In this problem, use $T_s = 0.5$.
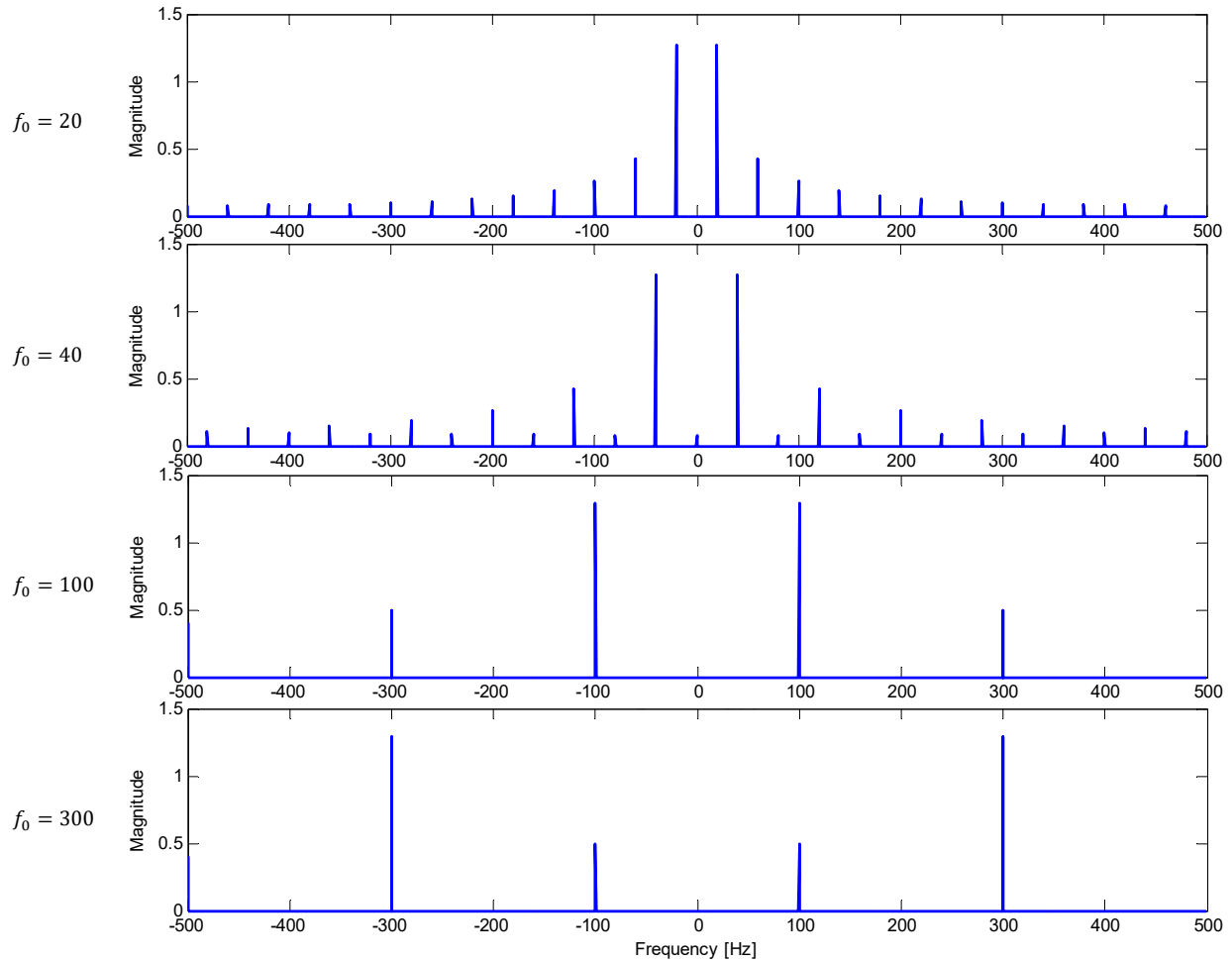
Figure 7.2:

(a) Use `MATLAB` to plot $\hat{g}(t)$ when only the $n = 0$ term is included.

(b) Use `MATLAB` to plot $\hat{g}(t)$ and all of its sinc components. Include only $n = -1, 0, 1$.

(c) Use `MATLAB` to plot $\hat{g}(t)$ and all of its sinc components.
Include only $n = -5, -4, \ldots, -1, 0, 1, \ldots, 4, 5$.

(d) Use `MATLAB` to plot $\hat{g}(t)$ and all of its sinc components.
Include only $n = -10, -9, \ldots, -1, 0, 1, \ldots, 9, 10$.

In all these plots, consider t from -4 to 4. Also include the plot of $\mathrm{sinc}(\pi t)$ for comparison.